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UNIFORM CONVEXITY OF UNITARY IDEALS

BY
NICOLE TOMCZAK-JAEGERMANN

ABSTRACT
If E is a symmetric Banach sequence which is q-concave with the constant equal to 1 (where $2 \leqq q < \infty$), then S_E is q-PL-convex. If E is q-concave and p-convex with the constants equal to 1 (where $1 < p \leqq 2 \leqq q < \infty$), then S_E is uniformly convex with modulus of convexity of power type q and uniformly smooth with modulus of smoothness of power type p.

In this note we continue study of geometric properties of unitary ideals of operators acting in a Hilbert space, started in the paper [1] jointly with D. J. H. Garling. Our main inequality for s-numbers of certain operators, stated in Proposition 1, allows us to use ideas developed in [1] in the context of p-convexity and q-concavity rather than K-(p,q)-monotonicity of symmetric Banach sequence spaces. As a result, we get information about original norms in unitary ideals, without need of a renorming.

Throughout the note we use standard notation from Banach space theory and from the theory of unitary ideals of operators acting in a Hilbert space. We refer to [3] for the definitions and notation from the theory of Banach lattices and to [2] for basic facts about operators acting in a Hilbert space.

If A is a compact operator acting in a Hilbert space then $|A|$ denotes the modulus of A, i.e. $|A| = \sqrt{A^*A}$ and $s(A) = \{s_j(A)\}_{j=1}^{\infty}$ denotes the sequence of singular numbers of A, i.e., $s_j(A)$ is the j-th eigenvalue of $|A|$, where eigenvalues are counted in nonincreasing order, according to their multiplicity. Suppose that E is a symmetric Banach sequence space (under the norm $\|\cdot\|$). The corresponding unitary ideal S_E is the space

$$S_E = \{A \text{ compact} : s(A) \in E\},$$

with the norm $\|A\|_E = \|s(A)\|$, for $A \in S_E$. In the case $E = l_p$, $1 \leqq p < \infty$, we use the notation $\|\cdot\|_p = \|\cdot\|_{l_p}$, and $\|\cdot\|_\infty = \|\cdot\|_{l_\infty}$ denotes the usual operator norm.

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Let us recall (cf. [1]) that a complex Banach space X is q-uniformly-PL-convex, where $2 \leqq q < \infty$, if there exist $c > 0$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \| x + e^{i\theta} y \| \, d\theta \geqq 1 + c \| y \|^q,$$

for every $x, y \in X$ with $\| x \| = 1$ and $\| y \| \leqq 1$.

Our results are based upon the following inequality for s-numbers of operators.

PROPOSITION 1. *Let $1 \leqq p \leqq 2$. Let A and B be compact operators. If A is positive and B is Hermitian then*

$$\sum_{j=1}^{k} s_j(A+iB)^p \leqq \sum_{j=1}^{k} s_j(A)^p + 2\sum_{j=1}^{k} s_j(B)^p, \tag{1}$$

for $k = 1, 2, \ldots$.

PROOF. Fix a positive integer k. Observe first that it is sufficient to consider the case when $\dim H \leqq 2k$. Indeed, let $\{e_j\}_{j=1}^k$ and $\{f_j\}_{j=1}^k$ be orthonormal systems in H such that $(A+iB)e_j = s_j f_j$ for $j = 1, \ldots, k$. Let $P: H \to H$ be the orthogonal projection onto $\operatorname{span}\{e_j, f_j\}_{j=1}^k$. Then $s_j(PAP + iPBP) = s_j(A+iB)$ for $j = 1, \ldots, k$, moreover, PAP is positive, PBP is Hermitian and $s_j(PAP) \leqq s_j(A)$ and $s_j(PBP) \leqq s_j(B)$ for $j = 1, \ldots, k$.

Let $n = \dim H$. Then Proposition 5 in [1] yields

$$\begin{aligned}
\sum_{j=1}^{n} s_j(A+iB)^p &= \| A + iB \|_p^p \\
&\leqq \| A \|_p^p + \| B \|_p^p \\
&= \sum_{j=1}^{n} s_j(A)^p + \sum_{j=1}^{n} s_j(B)^p.
\end{aligned} \tag{2}$$

LEMMA 1. *Let A and B be compact operators. If A is positive and B is Hermitian, then $s_j(A+iB) \geqq s_j(A)$ for $j = 1, 2, \ldots$.*

Assuming the truth of Lemma 1 we conclude the proof of Proposition 1 as follows:

$$\begin{aligned}
\sum_{j=1}^{k} s_j(A+iB)^p + \sum_{j=k+1}^{n} s_j(A)^p &\leqq \sum_{j=1}^{n} s_j(A+iB)^p \\
&\leqq \sum_{j=1}^{k} s_j(A)^p + \sum_{j=k+1}^{n} s_j(A) + \sum_{j=1}^{k} s_j(B)^p + \sum_{j=k+1}^{n} s_j(B)^p \\
&\leqq \sum_{j=1}^{k} s_j(A)^p + 2\sum_{j=1}^{k} s_j(B)^p + \sum_{j=k+1}^{n} s_j(A)^p.
\end{aligned}$$

Subtracting $\Sigma_{j=k+1}^{n} s_j(A)^p$ from both sides of the inequality, we get (1).

To prove Lemma 1 recall (cf. [2]) that for every compact operator T and for $j = 1, 2, \ldots$, one has

$$s_j(T) = \inf\{\| (I - Q)T \|_\infty \mid Q : H \to H \text{ is an orthogonal projection and rank } Q < j\}.$$

Fix j. Let $Q : H \to H$ be an orthogonal projection with rank $Q < j$. Put $P = I - Q$. Then P is an orthogonal projection too, in particular, $\| P \|_\infty = 1$ and $P = P^*$. Then

$$\begin{aligned} \| (I - Q)(A + iB) \|_\infty &\geqq \| P(A + iB)P \|_\infty \\ &\geqq \| PAP \|_\infty = \| P \sqrt{A} \|_\infty^2 \\ &= \| (I - Q) \sqrt{A} \|_\infty^2 \geqq s_j(\sqrt{A})^2 = s_j(A). \end{aligned}$$

Taking the infimum over all orthogonal projections Q with rank $Q < j$ we conclude that $s_j(A + iB) \geqq s_j(A)$.

Remark. An example given in [1] (the remark after Proposition 6) shows that the condition that A is positive cannot be dropped.

The next result shows that Proposition 1 has important consequences for unitary ideals. It also should be compared with Proposition 8 in [1].

Proposition 2. *Let E be a symmetric Banach sequence space.*

(i) *Let $1 < p \leqq 2$ and let E be p-convex with $M^{(p)}(E) = 1$. If A is a positive operator and B is a Hermitian operator in S_E, then*

$$\| A + iB \|_E^p \leqq \| A \|_E^p + 2 \| B \|_E^p. \tag{3}$$

(ii) *Let $1 < p \leqq 2 \leqq q < \infty$ and let E be p-convex and q-concave with $M^{(p)}(E) = 1 = M_{(q)}(E)$. Then there exist positive constants c and C, depending only on p and q, such that*

$$\begin{aligned} (\| A \|_E^q + c \| B \|_E^q)^{1/q} &\leqq [\tfrac{1}{2}(\| A + B \|_E^2 + \| A - B \|_E^2)]^{1/2} \\ &\leqq (\| A \|_E^p + C \| B \|_E^p)^{1/p} \end{aligned} \tag{4}$$

for all operators A and B in S_E.

Proof. (i) It is well known and easy to prove, applying Abel's transformation, that our assumption on E implies that there exists a set A of positive sequences such that

$$\| x \|^p = \sup_{a \in A} \sum_{n=1}^{\infty} a_n \Big(\sum_{j=1}^{n} x_j^{*p} \Big) \qquad \text{for } x \in E$$

(cf. [1] Proposition 10 and Theorem 3). Therefore,

$$\begin{aligned} \| A + iB \|_E^p &= \| \{ s_j(A + iB) \} \|^p \\ &= \sup_{a \in A} \sum_{n=1}^{\infty} a_n \sum_{j=1}^{n} s_j(A + iB)^p \\ &\leqq \sup_{a \in A} \sum_{n=1}^{\infty} a_n \sum_{j=1}^{n} s_j(A)^p + 2 \sup_{a \in A} \sum_{n=1}^{\infty} a_n \sum_{j=1}^{n} s_j(B)^p \\ &= \| \{ s_j(A) \} \|^p + 2 \| \{ s_j(B) \} \|^p \\ &= \| A \|_E^p + 2 \| B \|_E^p. \end{aligned}$$

(ii) Recall first that it follows from the result of G. Pisier [4] that there exists a symmetric Banach sequence space E_0 such that E is the complex interpolation space $[E_0, l_2]_\theta$, for some $0 < \theta < 1$ (in fact, $\theta = 2[1 - \max(1/p, (q-1)/q)]$). Therefore, $S_E = [S_{E_0}, S_{l_2}]_\theta$. Now, the observation of Pisier ([5]) yields that there exists $\alpha = \alpha(p, q) > 1$ such that

$$\| T + i\alpha R \|_E^2 + \| T - i\alpha R \|_E^2 \geqq \| T + R \|_E^2 + \| T - R \|_E^2, \tag{5}$$

for all operators T and R in S_E.

We pass now to the proof of the right-hand side inequality. Without loss of generality we may assume that A is a positive operator. Put $B_1 = \operatorname{Re} B$ and $B_2 = \operatorname{Im} B$. Then

$$\max(\| \alpha B_1 + B_2 \|_E, \| \alpha B_1 - B_2 \|_E) \leqq (1 + \alpha) \| B \|_E.$$

Therefore, by (3),

$$\begin{aligned} (\| A \|_E^p + 2(1+\alpha) \| B \|_E^p)^{1/p} &\geqq (\| A \|_E^p + \| \alpha B_1 + B_2 \|_E^p + \| \alpha B_1 - B_2 \|_E^p)^{1/p} \\ &\geqq \tfrac{1}{2}(\| A + i(\alpha B_1 + B_2) \|_E + \| A + i(\alpha B_1 - B_2) \|_E) \\ &\geqq [\tfrac{1}{2}(\| A + i(\alpha B_1 + B_2) \|_E^2 + \| A - i(\alpha B_1 - B_2) \|_E^2)]^{1/2}. \end{aligned}$$

Applying (5) for $T = A + iB_2$ and $R = B_1$, it follows that the last expression is equal to

$$\begin{aligned} [\tfrac{1}{2}(\| T + i\alpha R \|_E^2 + \| T - i\alpha R \|_E^2)]^{1/2} &\geqq [\tfrac{1}{2}(\| T + R \|_E^2 + \| T - R \|_E^2)]^{1/2} \\ &= [\tfrac{1}{2}(\| A + B_1 + iB_2 \|_E^2 + \| A - B_1 + iB_2 \|)_E^2]^{1/2} \\ &= [\tfrac{1}{2}(\| A + B \|_E^2 + \| A - B \|_E^2)]^{1/2}. \end{aligned}$$

The left-hand side inequality follows by a standard duality argument (cf. e.g. Proposition 4 in [1]) and we omit it.

Now we are ready to prove the main results of this note. They strengthen and complement the results of [1] (Theorem 4 and Theorem 5) and answer Problems 1, 4 and 5 raised there.

THEOREM 1. *Let $2 \leqq q < \infty$. Let E be a symmetric Banach sequence space which is q-concave with $M_{(q)}(E) = 1$. Then S_E is q-uniformly-PL-convex.*

PROOF. The dual E^* satisfies the assumptions of Proposition 2(i), with p such that $1/p + 1/q = 1$. Therefore the conclusion follows by an argument used in the proof of Theorem 1 in [1]; for completeness' sake we give a short proof. Let T, $R \in S_E$ with $\| T \|_E = 1$. Set $\beta = \| R \|_E^{q-1}/2^{2q-1}$, so that

$$1 + \beta \| R \|_E = [1 + 2(2\beta)^p]^{1/p}[1 + \| R \|_E^q/2^{2q-1}]^{1/q}.$$

There exists a partial isometry U such that UT is positive. Fix $\varepsilon > 0$. There exist operators A and B in S_{E^*}, with A positive and $\| A \|_{E^*} = 1 = \| B \|_{E^*}$ such that

$$\operatorname{trace} AUT \geqq 1 - \varepsilon, \qquad \operatorname{trace} BUR \geqq (1 - \varepsilon)\| R \|_E.$$

Let $C_\theta = ie^{i\theta}B^* - ie^{-i\theta}B$, for $0 < \theta \leqq 2\pi$. Since C_θ is Hermitian, it follows from Proposition 2(i) that

$$\| A + i\beta C_\theta \|_{E^*}^p \leqq 1 + 2(2\beta)^p.$$

Then

$$\begin{aligned}(1 - \varepsilon)(1 + \beta \| R \|_E) &\leqq \operatorname{Re} \operatorname{trace}(AUT + \beta BUR) \\ &= \operatorname{Re} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{trace}(A + i\beta C_\theta)(UT + e^{i\theta}UR)d\theta \\ &\leqq (1 + 2(2\beta)^p)^{1/p} \frac{1}{2\pi} \int_0^{2\pi} \| T + e^{i\theta}R \|_E d\theta.\end{aligned}$$

From the choice of β it follows, since ε is arbitrary, that

$$\frac{1}{2\pi} \int_0^{2\pi} \| T + e^{i\theta}R \|_E d\theta \geqq (1 + \| R \|_E^q 2^{2q-1})^{1/q}.$$

THEOREM 2. *Let $1 < p \leqq 2 \leqq q < \infty$. Let E be a symmetric Banach sequence space which is p-convex and q-concave with $M^{(p)}(E) = 1 = M_{(q)}(E)$. Then S_E is uniformly convex with modulus of convexity of power type q and uniformly smooth with modulus of smoothness of power type p.*

PROOF. Fix $\varepsilon > 0$. Let $T, R \in S_E$ with $\| T \|_E = 1 = \| R \|_E$ and $\| T - R \|_E = \varepsilon$. Applying the left-hand side inequality in (4) for $A = \frac{1}{2}(T + R)$, $B = \frac{1}{2}(T - R)$ one gets

$$(\| \tfrac{1}{2}(T + R) \|_E^q + c(\varepsilon/2)^q)^{1/q} \leqq 1.$$

So

$$1 = \| \tfrac{1}{2}(T + R) \|_E \geqq 1 - [1 - c\varepsilon^q/2^q]^{1/q}$$

$$\geqq c\varepsilon^q/q2^q.$$

Taking the infimum over all $T, R \in S_E$ such that $\| T \|_E = 1 = \| R \|_E$ and $\| T - R \|_E = \varepsilon$, it follows that $\delta_{S_E}(\varepsilon) \geqq c\varepsilon^q/q2^q$.

The proof for the modulus of smoothness, which uses the right-hand side inequality in (4), is similar and we omit it.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA T6G 2G1